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NUCLEAR PHYSICS B
Nuclear Physics B 744 [FS] (2006) 340–357

A new large N phase transition in YM_2

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Received 31 August 2005; received in revised form 22 February 2006; accepted 22 March 2006

Available online 6 April 2006

Abstract

Inspired by the interpretation of two-dimensional Yang–Mills theory on a cylinder as a random walk on the gauge group, we point out the existence of a large N transition which is the gauge theory analogue of the cutoff transition in random walks. The transition occurs in the strong coupling region, with the 't Hooft coupling scaling as $\alpha \log N$, at a critical value of α ($\alpha = 4$ on the sphere). The two phases below and above the transition are studied in detail. The effective number of degrees of freedom and the free energy are found to be proportional to $N^{2-\frac{\alpha}{2}}$ below the transition and to vanish altogether above it. The expectation value of a Wilson loop is calculated to the leading order and found to coincide in both phases with the strong coupling value.

1. Introduction

Yang–Mills theory in two dimensions is at the intersection of many different fields of theoretical physics. It is one example of non-trivial completely solvable gauge theory [1,2], in which both perturbative and non-perturbative effects can be studied. The interpretation of YM_2 in terms of string theories, namely a theory of branched coverings on a two-dimensional Riemann surface, was discovered for large N in [3], and for finite N in [4], for the intersecting Wilson loop on the plane. Its large N expansion has then been proved to describe a two-dimensional string theory on generic two-dimensional manifolds [5,6]. Non-trivial topological sectors in the unitary gauge also seem to be related to matrix string states [7]. Its partition function on a torus can be de-

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doi:10.1016/j.nuclphysb.2006.03.017

scribed in terms of a gas of free fermions [13–16], and the kernel on a cylinder by the evolution of a system of N free fermions on a circle, namely by the Sutherland model.

Some new connections between two-dimensional gauge theories and statistical mechanical systems were pointed out in [8] where two-dimensional gauge theories of the symmetric group S_n in the large n limit were investigated. Gauge theories of S_n also describe n-coverings of a Riemann surface and hence they are closely related to two dimensional Yang–Mills theories; the relation being essentially provided by Frobenius formula that relates $U(N)$ characters (in a representation with n boxes in the Young diagram) to the corresponding characters of the S_n group.

It was shown in [8] that the partition function of a gauge theory on a disc or a cylinder can be interpreted in terms of random walks on the gauge group, whose initial and final positions are the holonomies at the ends of the cylinder[1] and the number of steps is the area of the surface. Although the focus in [8] was on the discrete group S_n the argument can be trivially extended to continuous Lie groups. Similar results were independently obtained in [19–21]. It is well known in random walks theory that after a certain number of steps the end point of the walk becomes independent of the starting point: the walker has lost any memory of the point he started from. The critical number of steps after which that happens can be exactly calculated in a number of situations, and the corresponding transition is known as cutoff transition. Given the correspondence between random walks on the group and gauge theory on a cylinder, one expects to find the cutoff transition also in gauge theories. Indeed it was found in [8] that for an S_n gauge theory where the holonomy on each elementary plaquette is given by a single transposition,[2] a cutoff transition occurs in the large n limit when the number of plaquettes (and hence the area) is $\frac{1}{2}n\log n$, in agreement with previous results in random walks [17]. Models with more general Boltzmann weight for the plaquettes have a richer structure of phase diagrams [8]. The stringy interpretation of the cutoff transition in the S_n gauge theory is the following: beyond the transition the string world sheet is completely connected in the large n limit, while before the transition the world sheet consists of a large connected part and of a small fraction (in fact vanishing in the large n limit) of disconnected parts.

Another well known correspondence relates random walks with random graphs [18], that is graphs obtained by randomly connecting n points with p links. These can be put in correspondence with random walks on S_n made of p steps, each step consisting of a simple transposition. Two types of transitions are known in the large n large p double scaling limit in random graphs: a percolation transition at a critical value $\beta = \beta_c$ when $p = \beta n$ and the cutoff transition at $\alpha = 1/2$ when $p = \alpha n\log n$. Beyond the transition, namely for $p > 1/2n\log n$ all the n points are connected whereas before the transition a vanishing fraction of disconnected points survive.

It is rather natural at this point to look for a similar transition in the large N limit of $U(N)$ gauge theories. A large N phase transition on a sphere and on a cylinder in two-dimensional Yang–Mills theories—the Douglas–Kazakov phase transition—has been known for quite some time. However this is not a cutoff transition. In a cutoff transition the partition function on a disc for instance becomes independent on the holonomy on the border of the disc, and this is not the case in the Douglas–Kazakov transition. Besides the Douglas–Kazakov transition occurs at a finite value of the 't Hooft coupling whereas from the previous examples it appears that the cutoff transition occurs when the area scales as $\log N$ at large N. From this point of view the Douglas–

[1] In a disc the starting point is the identity of the group, and on a sphere both starting and ending points are the identity.

[2] In terms of the string interpretation this means that in each plaquette there is a single quadratic branch point connecting two of the n sheets of the world sheet.

Kazakov transition appears more similar to the percolation transition in random graphs, although a precise correspondence is still to be found.

The existence of the cutoff transition in the large N limit of 2D Yang–Mills on the sphere when the 't Hooft coupling scales as $\log N$ was proved by a simple argument in [8]. The present paper is devoted to study such transition further on both the sphere and the cylinder, in order to characterize its phases and give some physical interpretation. The paper is organized as follows: in Section 2 we review the large N Douglas–Kazakov transition and its physical interpretation. In Section 3 we introduce the cutoff transition on the sphere and study the phases above and below the transition. Section 4 is devoted to the transition on the disc and on the cylinder and Section 5 to the calculation of the expectation value of Wilson loops.

2. Large N transition

The partition function of a pure gauge theory on an arbitrary orientable two-dimensional manifold $\mathcal{M}$ of genus G, p boundaries and area $\tilde{A}$ has been known for many years [1,2]:

$$\begin{aligned} Z_{\mathcal{M}} &= \int [\mathcal{D}A^{\mu}] e^{-\frac{1}{4\lambda^2}\int_{\mathcal{M}} d^2x\,\sqrt{g}\,\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu})} \\ &= \sum_r \chi_r(g_1)\cdots\chi_r(g_p) d_r^{2-2G-p} e^{-\frac{A}{2N}C_2(r)}. \end{aligned} \tag{1}$$

The sum runs over all irreducible representations of the gauge group, λ is the gauge coupling, $\chi_r(g_i)$ is the character of the holonomy g_i in the representation r and $C_2(r)$ is the quadratic Casimir operator in the representation r. A is related to the actual area of $\mathcal{M}$ through $A = \lambda^2 N\tilde{A}$.

We consider $G = 0$ manifolds with at most 2 boundaries; i.e. spheres, discs and cylinders. Moreover, we will confine our analysis to the unitary groups $U(N)$ and $SU(N)$. A third order phase transition in the large N limit was discovered in the case of a sphere by Douglas and Kazakov [10] at a critical value $A = \pi^2$ of the rescaled area A. This transition appears to separate a weak coupling ($A < \pi^2$) from a strong coupling ($A > \pi^2$) regime. These results were generalized to the case of a cylinder in [11,12] where the phase transition was also interpreted as a result of instanton condensation.

The partition function on the sphere can be written as a sum over the set of integers $n_1 > n_2 > \cdots > n_N$ that label the irreducible representations of $SU(N)$ and $U(N)$[3]:

$$Z = e^{-\frac{A}{24}(N^2-1)} \sum_{n_1>n_2>\cdots>n_N} \prod_{i<j}(n_i - n_j)^2 e^{-\frac{A}{2N}\sum_{i=1}^{N}(n_i)^2}. \tag{2}$$

The existence of a phase transition at the critical value $A = \pi^2$ can be easily derived from (2) by noticing that the partition function (2) is exactly the same as the one of a Gaussian hermitian matrix model but with the integral over the eigenvalues replaced by the discrete sum over the integers n_i. The solution of a Gaussian hermitian matrix model in the large N limit is given by Wigner's semicircle distribution law for the eigenvalues[4]:

$$\left|\rho(\lambda)\right| = \frac{A}{2\pi}\sqrt{\frac{4}{A} - \lambda^2}, \tag{3}$$

[3] In the case of $U(N)$ the extra condition $n_N \geqslant -\frac{N-1}{2}$ must be imposed.

[4] This approach for the study of YM_2 was first used by Rusakov [9].

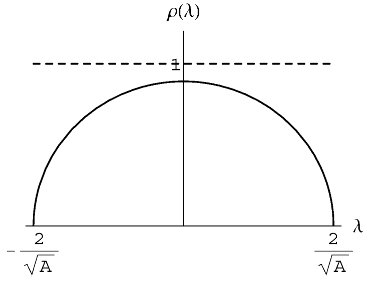

Fig. 1. Eigenvalue distribution for $A \leqslant \pi^2$.

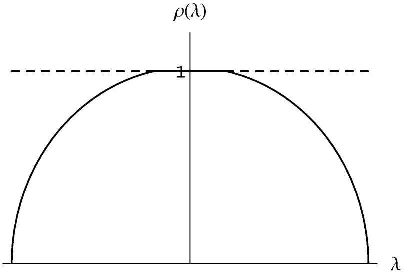

Fig. 2. Eigenvalue distribution for $A > \pi^2$ (i.e. DK transition).

where the continuum variables

$$x = \frac{i}{N}, \qquad \lambda(x) = \frac{n_i}{N},$$

and the corresponding density of eigenvalues

$$\rho(\lambda) = \frac{\partial x}{\partial \lambda} \tag{4}$$

have been introduced. In (2) however the would-be eigenvalues n_i are distinct integers and as a consequence the corresponding density $\rho(\lambda)$ in the large N limit is constrained by:

$$\left|\rho(\lambda)\right| \leqslant 1 \quad \forall \lambda. \tag{5}$$

Hence the Wigner semicircle solution is acceptable only in the weak coupling phase, namely for $A \leqslant \pi^2$ where the condition (5) is fulfilled. In fact the maximum of $|\rho(\lambda)|$ occurs at $\lambda = 0$, it increases with the area A and becomes equal to 1 at $A = \pi^2$, as easily seen from (3). The solution in the strong coupling phase $A > \pi^2$ was found in [10] and is expressed in terms of elliptic integrals. In this phase a finite fraction of the eigenvalues condenses, namely the distribution $\rho(\lambda)$ is flat and equal to one in a symmetric interval around $\lambda = 0$ as shown in Figs. 1 and 2.

2.1. Configuration space

It is well known [13–16] that the partition function of two-dimensional Yang–Mills theories with gauge group $U(N)$ can be interpreted in terms of a gas of N free fermions on a circle described by a Sutherland–Calogero model. In particular, if we denote by $\mathcal{K}_2(\theta, \phi; A)$ the kernel on a cylinder of scaled area A and with the $U(N)$ holonomies at the two ends given by the invariant angles θ_i and ϕ_i, it was shown that $\mathcal{K}_2(\theta, \phi; A)$ can be interpreted as the propagator in a time A from an initial configuration where the positions of the N fermions on the circle are given by θ_i and a final configuration with positions labeled by ϕ_i. The partition function on the sphere of area A is a particular case where the initial and final configurations are just $\theta_i = \phi_i = 0$ for all i's, namely the amplitude for a process where all fermions start at the origin and come back to the origin after a time A. By a modular transformation on the kernel of the cylinder one finds that the integers n_i labeling the irreducible representations of $U(N)$ are just the discrete momenta of the fermions on the circle. While in the momentum representation the Douglas–Kazakov phase transition can be interpreted as fermion condensation, in the configuration representation it can be seen in terms of instantons condensation [11,12]. In fact, while going from the initial $\theta_i = 0$ configuration to the final $\phi_i = 0$ configuration, a fermion can in principle wind an arbitrary number of times around the circle. These winding (instantons) configurations do not contribute in the large N limit to the weak coupling phase, as shown by the following simple argument. Consider the Wigner distribution (3) of momenta in the weak coupling phase. The maximum allowed momentum is $n_{\max} = \frac{2}{\sqrt{A}}$, hence the maximum shift in position for a single fermion in the time A is given by

$$\Delta\theta_{\max} = A\frac{2}{\sqrt{A}} = 2\sqrt{A}. \tag{6}$$

The existence of winding trajectories requires this shift in position to be at least 2π, namely A to be greater of π^2. Hence the critical value of A, where the Douglas–Kazakov phase transition occurs, marks the point where instantons condense, and contribute to the functional integral in the large N limit.

A more detailed understanding of the Douglas–Kazakov phase transition can be achieved by introducing in the large N limit the density $\rho(\theta, t)$ of fermions in the position θ at a given time (= area) t. Due to the compact nature of the configuration space $\rho(\theta, t)$ is defined in the interval $-\pi \leqslant \theta \leqslant \pi$ with $\rho(-\pi, t) = \rho(\pi, t)$. Matytsin proved [22] that if the evolution equation of the fermions is given by the Calogero–Sutherland model then the density $\rho(\theta, t)$ is governed by the Das–Jevicki equation [23] which admits Wigner semicircular distribution of radius $r(t)$ as a solution:

$$\rho(\theta, t) = \frac{2}{\pi r(t)^2}\sqrt{r(t)^2 - \theta^2}, \quad |\theta| \leqslant r(t) \tag{7}$$

with $r(t)$ satisfying the differential equation $\frac{d^2 r(t)}{dt^2} + \frac{4}{r(t)^3} = 0$. On a sphere of area A the boundary conditions are $r(0) = r(A) = 0$ and the differential equation has the solution:

$$r(t) = 2\sqrt{\frac{t(A-t)}{A}}. \tag{8}$$

The solution given by (7) and (8) is valid provided the support of the density function $\rho(\theta, t)$ is in the interval $[-\pi, \pi]$ at all t, namely provided $r(t) \leqslant \pi$. The maximum value for the radius $r(t)$ occurs for $t = A/2$ and is $r_{\max} = \sqrt{A}$. Hence the condition for the validity of the Wigner semicircular solution is $A \leqslant \pi^2$. Beyond the critical value $A = \pi^2$ the fermions "realize" that the space they live in is compact, instantons effects become important and (7) is not an acceptable solution any longer. On the cylinder a similar phase transition occurs in general, at a critical value of the area that depends on the holonomies at the boundaries.[5] If the distribution of the

[5] However with particular conditions at the boundary the phase transition may also be absent, see for instance [26].

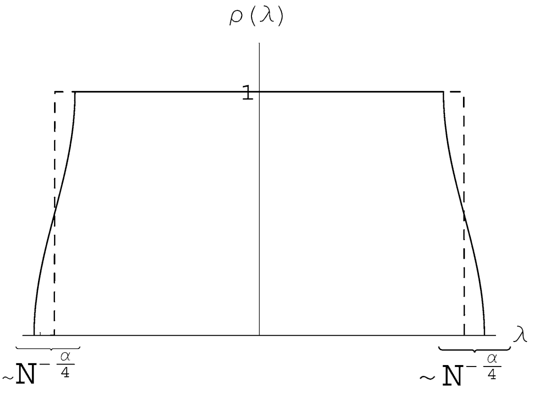

Fig. 3. Eigenvalues distribution for $0 < \alpha < 4$.

invariant angles of the holonomies at the boundaries is the Wigner semicircular distribution, then the critical area can be calculated exactly in the same way as for the sphere [11]. For a more general discussion see [12].

3. Large $A(N)$ phase transition

It is apparent from the discussion in the previous section and from the explicit form (2) of the partition function that as the area of the sphere increases the distribution of the "momenta" n_i (i.e. of the integers that label the $U(N)$ representations) becomes more and more similar to a double step function, like the one drawn with dashed lines in Fig. 3. In fact for very large areas the attractive quadratic potential tends to dominate over the repulsive force produced by the Vandermonde determinant. The double step distribution corresponds to the trivial representation of $U(N)$ in which all characters are identical irrespective of their argument. If this distribution dominates the functional integral then the kernel on a disc or on a cylinder becomes independent from the holonomies at the boundaries.

As already mentioned in the introduction, the partition functions on the disc and on the cylinder may be interpreted in terms of random walks on the group manifold. From this point of view the very large area phase, where the sum over the irreducible representations is dominated by the trivial representation corresponds to a walk which is so long that the walker has lost any notion of the starting point. The transition where this situation sets in is known in random walk theory as "cutoff transition", and the same term will be used here.

The cutoff regime occurs for areas larger than a critical N dependent value $A_c(N)$ which was found in [8]. The argument is very simple: consider the trivial double step representation R_0 that minimizes the Casimir term $\sum_i n_i^2$

$$R_0\colon\ \{n_1, \dots, n_N\} = \left\{\frac{N-1}{2}, \dots, -\frac{N-1}{2}\right\}, \tag{9}$$

and compare its contribution to the partition function (2) with the one coming from a representation R_1 in which n_1 has been increased by 1, namely in which $n_1 = \frac{N-1}{2} + 1$. In other words we look for the value of A at which R_0 ceases to be dominant. A simple calculation shows that the ratio between the two contributions is

$$\frac{Z_0(A,N,R_0)}{Z_0(A,N,R_1)}=\frac{e^{\frac{A}{2}}}{N^2}. \tag{10}$$

This ratio is larger than 1 (hence R_0 dominates and we are in the cutoff phase) if $A>4\log N$.

In order to study this phase transition in more detail, it is convenient to parametrize the area A by rescaling it with $\log N$:

$$A=\alpha\log N+\beta. \tag{11}$$

From the previous argument we expect the cutoff transition to occur at the critical value $\alpha_c=4$, separating two distinct phases. So Yang–Mills theory on a sphere seems to have four phases altogether: the first two at $\alpha=0$ separated by the Douglas–Kazakov phase transition at $\beta=\pi^2$, the other two when the area is logarithmically rescaled with N that are separated by the cutoff transition. The phase before the cutoff is qualitatively different from the strong coupling phase à la Douglas–Kazakov, as we will show in the following. Nonetheless, since the rescaling of the area with $\log N$ is essentially introduced by hand, it is hard to say whether there is some higher-order phase transition between them or just a crossover.

The aim of this section is to find the saddle point configuration and the free energy in the two phases above and below the cutoff point $\alpha_c=4$: the cutoff phase $\alpha>\alpha_c$ has been discussed above and it is rather trivial, but the phase below α_c appears as an interesting intermediate phase between the strong coupling phase in the Douglas–Kazakov transition and the cutoff phase. Hence we shall concentrate on this in the rest of the section.

Let us consider again the partition function (2) and the corresponding action

$$S=2\sum_{i>j=1}^{N}\log|n_i-n_j|-\frac{A}{2N}\sum_{i=1}^{N}n_i^2. \tag{12}$$

We want to find the extremum of this action in the large N limit, when A is parametrized as in Eq. (11). Since we expect the saddle point distribution of the "momenta" n_i to be symmetric with respect to the origin ($n_i\to-n_i$) we shall perform the variation only with respect to symmetric configurations, that is we set

$$-M\leqslant i\leqslant M,\quad \text{with } M=\frac{N-1}{2} \text{ and } n_{-i}=-n_i. \tag{13}$$

Using this symmetry one can restrict the sums to non-negative values of i ($i=1,\dots,M$) and write the action as:

$$S=2\sum_{i>j\geqslant1}^{M}\log(n_i-n_j)^2+\sum_{i>j\geqslant1}^{M}\log(n_i+n_j)^2-\frac{A}{2M}\sum_{i=1}^{M}n_i^2. \tag{14}$$

While in the cutoff phase $n_i=i$ for $i=1,\dots,M$ below the cutoff transition we expect a configuration of the type described in Fig. 3, namely:

$$\begin{aligned}&n_i=i,\quad i=1,\dots,M-l,\\&n_{M-l+\alpha}=M-l+\alpha+r_\alpha,\quad \alpha=1,\dots,l,\end{aligned} \tag{15}$$

where the value of l and the spectrum of the integers r_α are to be determined. With these notations the action can be written as:

$$S - S_0 = 4\sum_{\alpha=1}^{l}\sum_{j=1}^{M-l}\log\left[\left(1+\frac{r_\alpha}{\alpha+j}\right)\left(1+\frac{r_\alpha}{M-l+\alpha+j}\right)\right] + 2\sum_{\alpha\neq\beta}\left(\log\left(1+\frac{r_\alpha+r_\beta}{\alpha+\beta+2(M-l)}\right)+\log\left(1+\frac{r_\alpha-r_\beta}{\alpha-\beta}\right)\right) - \frac{A}{2M}\sum_{\alpha=1}^{l}\left(2(M-l+\alpha)r_\alpha+r_\alpha^2\right), \tag{16}$$

where S_0 represents the value of the action in the trivial representation R_0. We shall assume that as $M\to\infty$ also $l\to\infty$ but at a slower rate than M, namely $l/M\to 0$. We shall also assume that l and r_α will be of the same order in the large M limit. These assumptions will be justified a posteriori, in the sense that they will provide a stable saddle point in the large M limit when the area is scaled like $\log M$. They are also very reasonable assumptions: l is of order N in the strong coupling phase following the Douglas–Kazakov transition and one expects that with the logarithmic rescaling of the area it will shrink further by some power of N.

The first step in dealing with (16) is to make the dependence from M explicit. By using the identity

$$\prod_{j=1}^{M-l}\left(1+\frac{r_\alpha}{j+\alpha}\right)\left(1+\frac{r_\alpha}{M-l+j+\alpha}\right) = \frac{(2M-2l+\alpha+r_\alpha)!}{(2M-2l+\alpha)!}\frac{\alpha!}{(\alpha+r_\alpha)!} \tag{17}$$

we can rewrite the action as

$$S - S_0 = 4\sum_{\alpha=1}^{l}\log\frac{(2M-2l+\alpha+r_\alpha)!}{(2M-2l+\alpha)!}\frac{\alpha!}{(\alpha+r_\alpha)!} + 2\sum_{\alpha\neq\beta}\left(\log\left(1+\frac{r_\alpha+r_\beta}{\alpha+\beta+2(M-l)}\right)+\log\left(1+\frac{r_\alpha-r_\beta}{\alpha-\beta}\right)\right) - \frac{A}{2M}\sum_{\alpha=1}^{l}\left(2(M-l+\alpha)r_\alpha+r_\alpha^2\right). \tag{18}$$

All ratios of factorials in (18) can be reduced to the form $\log\frac{(N+C)!}{N!}$ with $N\to\infty$, $C\to\infty$ and $\frac{C}{N}\to 0$. By repeated use of Stirling formula one finds, up to terms that vanish as $\frac{C}{N}\to 0$:

$$\log\frac{(N+C)!}{N!} \sim C\left[\log N + f\left(\frac{C}{N}\right)\right] \tag{19}$$

where

$$f\left(\frac{C}{N}\right) = \log\left(1+\frac{C}{N}\right)+\frac{\log(1+\frac{C}{N})}{\frac{C}{N}} - 1 = \sum_{k=1}^{M-l}(-1)^{k-1}\frac{z^k}{k(k+1)}. \tag{20}$$

By using this asymptotic behaviour, and introducing continuum variables in the large N limit, namely

$$x=\frac{\alpha}{l}\quad r(x)=\frac{r_\alpha}{l}\quad \sum_\alpha = l\int dx. \tag{21}$$

The action finally takes the form

$$\begin{aligned}S-S_0 = 4l^2\int_0^1 dx\Big[r(x)\Big(\log\Big(\frac{M}{l}\Big)-\log\big(x+r(x)\big)+1+\log 2\Big)\\
+x\big(\log x-\log\big(x+r(x)\big)\big)\Big]+2l^2\int_0^1 dx\int_0^1 dy\log\Big(1+\frac{r(x)-r(y)}{x-y}\Big)\\
-Al^2\int_0^1 dx\, r(x),\end{aligned} \tag{22}$$

where subleading terms (by powers of $\frac{l}{M}$) have been neglected.

It is apparent from (22) that in the new regime $M\gg l\gg 1$ the quadratic term coming from the Casimir operator has been linearized. Thus our problem reduces to a particular case of a class of models characterized by a repulsive Vandermonde-like term and an attractive linear term, which have been widely studied in the literature starting from [24] (see also [25] and references therein). By the very same procedure showed at the beginning of the present section, Eqs. (9), (10), one can prove that a cutoff transition exists also for general linear models.

Let us now parametrize the area A according to Eq. (11) and write the action as:

$$S-S_0=4l^2[F_0\log M-F_1\log l+F_2], \tag{23}$$

where F_0, F_1 and F_2 are of order 1 in the large M and l limit and are given by:

$$\begin{aligned}F_0&=\Big(1-\frac{\alpha}{4}\Big)\int_0^1 dx\, r(x),\\
F_1&=\int_0^1 dx\, r(x),\\
F_2&=\int_0^1 dx\Big[r(x)\Big(-\log\big(x+r(x)\big)+1-\log 2-\frac{\beta}{4}\Big)+x\big(\log x-\log\big(x+r(x)\big)\big)\Big]\\
&\quad+\frac{1}{2}\int_0^1 dx\int_0^1 dy\log\Big(1+\frac{r(x)-r(y)}{x-y}\Big).\end{aligned}$$

In order to find the configuration that maximizes the functional integral in the large M limit we take the variation of (23) with respect to both l and $r(x)$. The variation with respect to l gives the equation:

$$\left(1-\frac{\alpha}{4}\right)\log M+\left(\frac{F_2}{F_1}-\frac{1}{2}\right)=\log l \tag{24}$$

which shows that l grows like $M^{1-\frac{\alpha}{4}}$. This is consistent with what we expected: for $\alpha\to 0$ it gives $l\sim M$ as in the strong phase beyond the Douglas–Kazakov transition, and at the cutoff point $\alpha=4$ the power vanishes as expected.

The variation with respect to $r(x)$ gives on the other hand

$$\int_0^1 dy\frac{1}{x+r(x)-y-r(y)}-\log\bigl(x+r(x)\bigr)=C \tag{25}$$

with

$$C=\log l-\left(1-\frac{\alpha}{4}\right)\log M+\frac{\beta}{4}+\log 2=\frac{F_2}{F_1}-\frac{1}{2}+\frac{\beta}{4}+\log 2. \tag{26}$$

If one introduces the new variable $\xi=x+r(x)$ and the density function $\rho(\xi)=\frac{dx}{d\xi}$ with support in the interval $[0,a]$, Eq. (25) becomes[6]:

$$\int_0^a d\eta\frac{\rho(\eta)}{\xi-\eta}-\log\xi=C. \tag{27}$$

This is a standard type of equation for the density of eigenvalues in the large N limit of matrix models and can be solved by standard analytic methods (for a detailed discussion of this equation see for instance [27]).

This same equation was obtained in [25], Eq. (66), for the large-N limit of models with a linear potential. The only seeming difference is the presence, in our case, of a logarithmic non-constant term; this is because we have explicitly performed the integral in the region where $\rho=1$.

The resolvent function, whose discontinuity across the cut gives the density $\rho(\xi)$, is given by:

$$H(\xi)=\log\xi+C-2\log\left(\frac{\sqrt{\xi-a}+\sqrt{\xi}}{\sqrt{a}}\right) \tag{28}$$

with the additional condition that for large ξ

$$H(\xi)=\frac{1}{\xi}+\mathcal{O}\left(\frac{1}{\xi^2}\right). \tag{29}$$

The corresponding density is given by

$$\rho(\xi)=\frac{2}{\pi}\arccos\left(\sqrt{\frac{\xi}{a}}\right). \tag{30}$$

This solution obviously describes, through the symmetry (13), both the positive and the negative region of n_i.

[6] The lowest extreme of the interval is $r(0)$ which is zero by construction, the upper end a is, according with the definition of ξ, $a=1+r(1)$.

The asymptotic condition (29) gives the two extra equations

$$a = 2, \qquad C = \log 2. \tag{31}$$

Eq. (30), together with the first of (31) define $r(x)$ completely, although in an implicit way. Hence all integrals involved in the definition of F_1 and F_2 can be calculated. The calculation can actually be done analytically and gives:

$$F_1 = \frac{1}{4}, \qquad \frac{F_2}{F_1} = \frac{1}{2} - \frac{\beta}{4}. \tag{32}$$

The second equation could have been derived independently from (26) and (31), so it constitutes a non trivial consistency check. We can now write explicitly l and the free energy F in terms of the area $A(M)$:

$$\begin{aligned} l &= e^{-\frac{\beta}{4}} M^{1-\frac{\alpha}{4}} = M e^{-\frac{A(M)}{4}}, \\ F &= S - S_0 = \frac{1}{2} l^2 = \frac{1}{2} e^{-\frac{\beta}{2}} M^{2-\frac{\alpha}{2}} = \frac{1}{2} M^2 e^{-\frac{A(M)}{2}}. \end{aligned} \tag{33}$$

Beyond the cutoff transition we have instead $F = l = 0$, as the dominant eigenvalue distribution is given by the double step function sketched in Fig. 3.

The interpretation of these results from the point of view of the free fermion description is very clear: beyond the cutoff ($\alpha > 4$) we are effectively in a zero temperature situation where all fermions fill the Fermi sea with no holes. Below the cutoff instead ($\alpha < 4$) some excited fermions and the corresponding holes are present in proximity of the surface of the Fermi sea both on the positive and negative momentum side. The number of fermions above the sea level is given by l in (33). The ratio $\frac{l}{M}$ vanishes like $M^{-\frac{\alpha}{4}}$ in the large M limit. This distinguishes this phase from the strong coupling phase of the Douglas–Kazakov transition, where such ratio remains finite, namely the number of fermions above the Fermi sea level is of order M.

In spite of being described in terms of N free fermions, the free energy is proportional (with the standard 't Hooft scaling) to N^2, which reflects the original number of degrees of freedom in a unitary $N \times N$ matrix model.[7] So it is not surprising that the free energy becomes proportional to l^2 in presence of l effective fermionic degrees of freedom when the area is rescaled by a $\log M$ factor. It is as if the effective size of the original matrix had shrunk to $l \times l$. A full understanding of the reduction of number of degrees of freedom from the point of view of the original gauge degrees of freedom is still wanted, although some light on it might be thrown by the study of the kernels on the disc and the cylinder in the following sections.

It is apparent from (33) that the number of effective degrees of freedom $M^{2-\frac{\alpha}{2}}$ is the actual order parameter for the cutoff transition. Incidentally, this unusual dependence of the number of degrees of freedom upon α, together with the choice of a large M limit, makes it rather delicate to classify such a transition according to standard terminology.

Let us finally consider the representations of $U(N)$ and/or $SU(N)$ that correspond to the "momentum" distribution pictured in Fig. 3 and given in (30). With the group $SU(N)$[8] this corresponds to a composite representation in the sense of Ref. [6], whose Young diagram is shown

[7] The description in the terms of N fermions follows the integration over the angular variables that reduces the matrix model to an integral over the eigenvalues. The ensuing Vandermonde determinant makes the wave function describing the eigenvalues antisymmetric. As a consequence the total momentum of M left moving fermions is of order M^2 rather than M, that is of the same order as the original number of bosonic degrees of freedom.

[8] With $SU(N)$ the term $\sum_i n_i^2$ in the action should be replaced by $\sum_i n_i^2 - \frac{1}{N}(\sum_i n_i)^2$ which is invariant under $n_i \to n_i + a$. However with suitable choice of a the extra term can be set equal to zero.

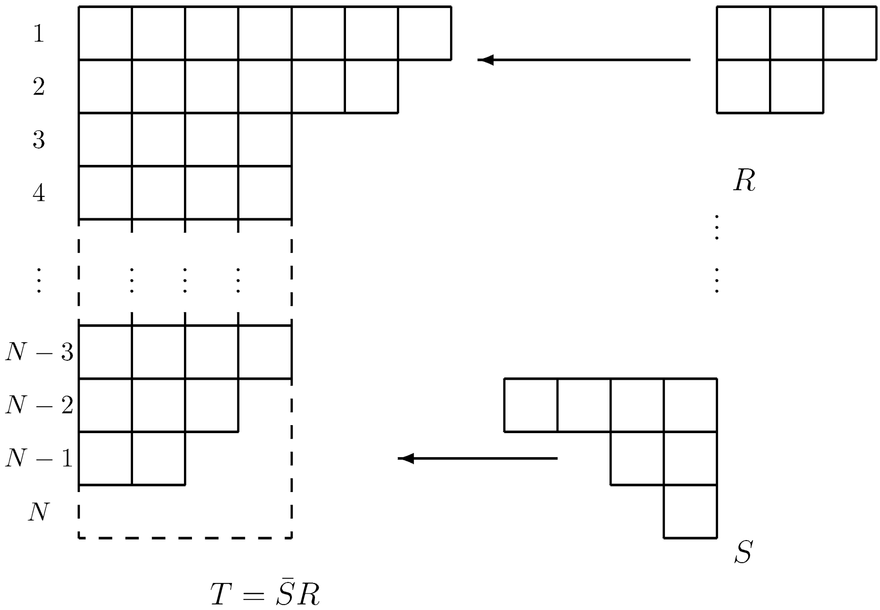

Fig. 4. Young tableau for a composite representation.

in Fig. 4 where the constituent representations are denoted by R and S. The rows in the Young diagram of R and S (that coincide in this case) have lengths r_α and their total number of boxes $|r|$ is given in the large N limit by

$$|r| = \sum_\alpha r_\alpha \sim l^2 \int dx\, r(x) = \frac{1}{4} l^2. \tag{34}$$

If the group is U(N) an arbitrary number of columns of length N can be added or subtracted to the Young diagram of Fig. 4, and the composite representation can be seen as the direct product of the two constituent representations R and S with opposite $U(1)$ charges. In fact the integers labeling the representation S are in this case negative, which corresponds to changing signs of the invariant angles θ_i, namely to changing U into $U^\dagger$. In the strong coupling regime of the Douglas–Kazakov transition ($A > \pi^2$) the partition function is dominated in the large N limit by a composite representation of the same type but with the Young diagram of R and S made of rows and columns of lengths of order N, rather than l, and a total number of boxes of order N^2 rather than l^2.

4. Cutoff transition on the disc and on the cylinder

In this section we are going to consider the partition function on a disc and on a cylinder, with fixed holonomies at the boundaries, in the regime where the area A is scaled as in (11). We shall show that the cutoff transition occurs also in this case at the same critical value of α, except for some particular holonomies at the boundaries. This is in analogy to what happens in the case of the Douglas–Kazakov transition which was proved to occur on a cylinder in [11,12], except possibly for some special configurations (see [26]).

We shall use the standard expression for the partition function on a cylinder with holonomies U and V at the boundaries, which is a particular case of (1):

$$Z_{\mathrm{cyl}}(U,V)=e^{-\frac{A}{24}(N^2-1)}\sum_r \chi_r(U)\chi_r(V)e^{-\frac{A}{2N}\sum_i n_i^2}. \tag{35}$$

The partition function on a disc is obtained from (35) by taking for instance $U\to 1$ and the partition function on the sphere is recovered by taking the double limit $U\to 1$ and $V\to 1$. The area A in (35) scales as in (11). The sum over the representations r is replaced in the large N limit by the saddle point, namely by a representation whose "momentum" distribution is of the type described in Fig. 3. This corresponds to a composite representation, whose constituent representations R and S have rows and columns of order l with $l/N\to 0$ in the large N limit. The first step in evaluating (35) is then to calculate the characters in such limit.

4.1. Characters in the large N, large l limit

For simplicity, let us consider first the case where only one constituent representation is present, which would be the case if only right (or left) moving fermions were present above the Fermi sea. This is given by:

$$\begin{aligned} &n_i=i \quad \text{for } i=1,\dots,N-l,\\ &n_i=N-l+\alpha+r_\alpha \quad \text{for } i=N-l+\alpha \text{ and } \alpha=1,\dots,l. \end{aligned} \tag{36}$$

The dimension Δ_r of this representation can be written in the form

$$\begin{aligned} \Delta_r&=\prod_{i>j}\frac{|(n_i-n_j)|}{(i-j)}=\prod_\alpha\frac{\alpha!}{(\alpha+r_\alpha)!}\prod_{\alpha>\beta}\left(1+\frac{r_\alpha-r_\beta}{\alpha-\beta}\right)\prod_\alpha\frac{(N-l+\alpha+r_\alpha)!}{(N-l+\alpha)!}\\ &=\frac{d_r}{|r|!}\prod_\alpha\frac{(N-l+\alpha+r_\alpha)!}{(N-l+\alpha)!}=\frac{d_r}{|r|!}N^{\sum r_\alpha}\left[1+\mathcal{O}\left(\frac{l}{N}\right)\right], \end{aligned} \tag{37}$$

where d_r is the dimension of the representation of the symmetric group $S_{|r|}$ associated to the Young diagram r of rows r_α. The order of the symmetric group is the total number of boxes in the Young diagram: $|r|=\sum_\alpha r_\alpha$. The dependence on N in (37) is explicit: the ratio of factorials in (37) is a polynomial in N of degree $|r|$. If l, α and r_α are all of order l the coefficient of $N^{|r|-k}$ in this polynomial is of order l^k in the large N, large l limit. Hence we can write

$$\Delta_r=\frac{d_r}{|r|!}N^{|r|}\left[1+\sum_k^{|r|}c_k\left(\frac{l}{N}\right)^k\right], \tag{38}$$

where the coefficients c_k are smooth in the large l, large N limit. If the limit is taken keeping l/N finite, as in the strong coupling phase of the Douglas–Kazakov transition, all terms at the r.h.s. of (38) are of the same order and cannot be neglected. On the other hand if the double scaling limit is taken with $l/N\to 0$, as in the previous section, then all terms after the 1 are subleading and can be neglected.

The same argument holds if instead of the dimension of the representation we consider a character of $U(N)$. In fact the celebrated Frobenius formula gives:

$$\chi_r(U)=\frac{d_r}{|r|!}\sum_{\sigma\in S_{|r|}}\frac{\hat{\chi}_r(\sigma)}{d_r}N^{k_\sigma}\prod_j\left(\frac{\operatorname{Tr}U^{s_j}}{N}\right), \tag{39}$$

where $\hat{\chi}_r(\sigma)$ is a character of the symmetric group $S_{|r|}$ in the representation labeled by the same Young diagram as $\chi_r(U)$, s_j are the lengths of the cycles in the cycle decomposition of σ ($\sum_j s_j = |r|$) and k_σ is the number of cycles in σ. By taking $U \to 1$ and comparing (38) and (39) one finds that the contribution of $\frac{\hat{\chi}_r(\sigma)}{d_r}$ when σ consists of k_σ cycles is $\sim l^{|r|-k_\sigma}$ in the large l limit. So if we take the double scaling limit where both l and N go to infinity and the ratio l/N goes to zero, all terms in the sum over σ in (39) are subleading with respect to the one where σ is the identity. We obtain:

$$\chi_r(U) = \frac{d_r}{|r|!}\left(\frac{\operatorname{Tr} U}{N}\right)^{|r|} N^{|r|}\big[1+\mathcal{O}(l/N)\big] = \Delta_r \left(\frac{\operatorname{Tr} U}{N}\right)^{|r|}\big[1+\mathcal{O}(l/N)\big]. \tag{40}$$

Notice again that in the strong coupling phase of the Douglas–Kazakov transition, where in the large N limit l/N is kept constant, all terms in (39) coming from different permutations σ are of the same order and cannot be neglected.

4.2. Partition function on the cylinder and special boundary conditions

As a result of previous analysis we find that in the composite representations that are dominant in the region $0 < \alpha < 4$ the characters $\chi_r(U)$ depend only from $\operatorname{Tr} U$. Hence the partition function on the cylinder will depend only on the trace of the holonomies on the boundaries. In fact it is apparent from (40) that going from the sphere ($U = 1$) to the cylinder just amounts to a multiplicative factor $(\frac{\operatorname{Tr} U}{N})^{|r|}$. In the case of interest however the composite representation contains both chiral and anti-chiral component representations (that is R and S of Fig. 4) and not just one as in the simplified example discussed above. However in the large N limit the two component representations are decoupled [6] and the character of the composite representation becomes just the product of the characters of the component representations:

$$\chi_{\text{composite}}(U) = \chi_r(U)\chi_r\big(U^\dagger\big) = \Delta_r^2\left(\left|\frac{\operatorname{Tr} U}{N}\right|\right)^{2|r|}. \tag{41}$$

By replacing (41) into (35) we find

$$Z_{\text{cyl}}(U,V) = Z_{\text{sphere}}\left(\left|\frac{\operatorname{Tr} U}{N}\frac{\operatorname{Tr} V}{N}\right|\right)^{2|r|}, \tag{42}$$

where of course the value of $|r|$ is the one determined by the saddle point equations and the equality holds, in the large N limit, only in the regime where the area A is scaled as in (11) with $\alpha \neq 0$. If we set $u = |\frac{\operatorname{Tr} U}{N}|$ and $v = |\frac{\operatorname{Tr} V}{N}|$ it is almost immediate to see that the multiplicative factor at the r.h.s. of (42) is equivalent to replace in the action (23) the constant term β in the area A with a $\hat{\beta}(u,v)$ given by:

$$\beta \to \hat{\beta}(u,v) = \beta - \log u^2 - \log v^2. \tag{43}$$

The partition function on the cylinder is then the same as the partition function on a sphere whose area is obtained from the area of the cylinder by adding the two terms $\log u^2$ and $\log v^2$. The latter can be interpreted as the areas of the two discs necessary to go, in the given momentum configuration, from $U = 1$ (respectively $V = 1$) to the holonomy at the boundary with $|\frac{\operatorname{Tr} U}{N}| = u$ (respectively $|\frac{\operatorname{Tr} V}{N}| = v$). The areas of the two discs are of order 1, so this correction does not affect the position of the cutoff transition that remains on the cylinder at $\alpha = 4$.

The discussion above relies on the fact that the leading term in Frobenius formula (39) comes from the identical permutation. However this is not always true: if we take the large N and l limit keeping $\mathrm{Tr}\,U = 0$[9] then all terms in (39) with σ containing cycles of length 1 would vanish. Assuming that $\mathrm{Tr}\,U^2 \neq 0$, the term in Frobenius formula (39) with the highest power on N would then come from permutations σ *all* made out of cycles of length 2 and would be of order $N^{\frac{|r|}{2}}$ instead of $N^{|r|}$. Supposing that both the trace of U and of V vanish the coefficient of $\log M$ in the first term of (22) would be halved. Correspondingly the critical value of α at which the cutoff transition occurs would also be halved and become $\alpha_c = 2$.

Let us make this argument general and more quantitative. Let us assume that $\mathrm{Tr}\,U^{k_1} \neq 0$ with $\mathrm{Tr}\,U^j = 0$ for $j < k_1$ and the same for V with k_2 at the place of k_1. The leading term in (39) will now be:

$$\chi_r(U) = \frac{d_r}{|r|!}\left(\frac{\mathrm{Tr}\,U^{k_1}}{N}\right)^{\frac{|r|}{k_1}} \frac{\hat{\chi}_r(\sigma)}{d_r} N^{\frac{|r|}{k_1}}, \tag{44}$$

where σ consists of $\frac{|r|}{k_1}$ cycles of length k_1.[10] From the discussion following (39) we resume that in the large l limit $\frac{\hat{\chi}_r(\sigma)}{d_r} \sim l^{|r|(1-\frac{1}{k_1})}$. By using this asymptotic behaviour and Eq. (44) we can write the partition function on the cylinder in the large N and l limit as in (23), but with F_0 and F_1 replaced by the following expressions:

$$F_0 = \left(\frac{1}{2k_1} + \frac{1}{2k_2} - \frac{\alpha}{4}\right)\int_0^1 dx\, r(x),$$

$$F_1 = \left(\frac{1}{2k_1} + \frac{1}{2k_2}\right)\int_0^1 dx\, r(x). \tag{45}$$

The dependence of F_2 from $r(x)$ is modified, in a so far unknown way, by next to leading terms in $\frac{\hat{\chi}_r(\sigma)}{d_r}$[11] while the constant parameter β is replaced by

$$\beta \to \hat{\beta}(u_{k_1}, v_{k_2}) = \beta - \frac{1}{k_1}\log u_{k_1}^2 - \frac{1}{k_2}\log v_{k_2}^2, \tag{46}$$

where, with obvious notations, $u_{k_1} = \frac{|\mathrm{Tr}\,U^{k_1}|}{N}$ and $v_{k_2} = \frac{|\mathrm{Tr}\,V^{k_2}|}{N}$. Although the equation for $r(x)$ cannot be derived, the variation with respect to l gives the scaling power of l and the cutoff transition point:

$$l \sim M^{1-\frac{\alpha}{4(\frac{1}{2k_1}+\frac{1}{2k_2})}}. \tag{47}$$

The cutoff transition occurs then at critical point $\alpha_c(k_1, k_2)$ given by:

$$\alpha_c(k_1, k_2) = 4\left(\frac{1}{2k_1} + \frac{1}{2k_2}\right) \tag{48}$$

[9] This can happen for instance if the holonomy at a boundary has a symmetry of some sort, for instance of the type $\theta \to \theta + \pi$, that is preserved through the limiting process.

[10] We assume here for simplicity that we take $l \to \infty$ keeping the total number of boxes in the Young diagram multiple of k_1 and k_2.

[11] A lot is known on the characters of permutations with cycles all of the same length, so an explicit expression for F_2 is probably obtainable.

which generalizes the result on the sphere.

5. Wilson loops

In this section we calculate the expectation value of a Wilson loop in a Yang–Mills theory with gauge group $U(N)$ in the large N limit. The space–time manifold has the topology of a two-dimensional sphere whose area scales like $\log N$ as in Eq. (11). We shall follow th approach of Daul and Kazakov [29] and Boulatov [30] who did the calculation for constant areas. The general solution for arbitrary self-intersecting Wilson loops was obtained in [28].

We may think of the sphere of area A as two discs of areas A_1 and A_2 ($A_1 + A_2 = A$) sewed along their common boundary and with holonomy on the boundary respectively U and $U^\dagger$. The Wilson loop is then given by

$$\begin{aligned} W(A_1, A_2) &= \left\langle \frac{1}{N} \operatorname{Tr} U \right\rangle \\ &= \frac{1}{Z} \sum_{R_1, R_2} d_1 d_2 \int dU \frac{1}{N} \operatorname{Tr} U \chi_1(U) \chi_2\big(U^\dagger\big) e^{-\frac{A_1}{2N} C_1 - \frac{A_2}{2N} C_2}, \end{aligned} \tag{49}$$

where d_1 and C_1 are the dimension and the Casimir operator in the representation R_1, referred to the disc of area A_1; likewise for d_2 and C_2. The quantity $\int dU \operatorname{Tr} U \chi_1(U) \chi_2(U^\dagger)$ may be either 0 or 1, namely it is 1 when the Young diagram of R_2 is obtained by adding one box to the diagram of R_1 and 0 otherwise. That is, if R_1 is labeled by the integers $n_1 > n_2 > \cdots > n_N$, R_2 is labeled by a set on integers where one of the n_i is increased by one. Daul and Kazakov used this property to get rid of one summation and obtained

$$W(A_1, A_2) = \frac{1}{Z} \sum_{R_1} \frac{1}{N} \sum_i d_1^2 \prod_{j, j \neq i} \left(1 + \frac{1}{n_j - n_i}\right) e^{-\frac{A_1 + A_2}{2N} C_1} e^{\frac{A_2}{N} n_i}, \tag{50}$$

where the sum over i corresponds to all the possible ways of adding one box to the diagram. This is not however the whole result, in fact the original expression is symmetric under exchange of A_1 and A_2, so a term with A_1 and A_2 exchanged[12] must be added to (50) and gives:

$$W(A_1, A_2) = \frac{1}{Z} \sum_{R_1} \frac{1}{N} \sum_i d_1^2 \prod_{j, j \neq i} \left(1 + \frac{1}{n_j - n_i}\right) e^{-\frac{A_1 + A_2}{2N} C_1} \big(e^{\frac{A_1}{N} n_i} + e^{\frac{A_2}{N} n_i}\big). \tag{51}$$

Moreover, Eq. (50), as well as (51), is clearly not symmetric under $n_i \leftrightarrow -n_i$, so we cannot restrict our considerations to $n_i > 0$ any longer; instead we have to consider the whole interval $-\infty < n_i < \infty$.

Let us first compute $W(A_1, A_2)$ in the *frozen* phase where the sum over R_1 is dominated by the trivial representation of dimension $d_1 = 1$ labeled by $n_i = i - \frac{N+1}{2}$ $\forall i$ and R_2 is the fundamental representation of dimension $d_2 = N$ with $C_2 - C_1 = N$. By inserting this into (51) one finds:

$$W(A_1, A_2) = \big(e^{-\frac{A_1}{2}} + e^{-\frac{A_2}{2}}\big) \tag{52}$$

which is, as expected, a typical strong coupling result. If for instance $A_1 \gg A_2$, then we have

$$W(A_1, A_2) \sim e^{-\frac{A_2}{2}}. \tag{53}$$

[12] This term originates from the fact that $\int dU \operatorname{Tr} U \chi_1(U) \chi_2(U^\dagger)$ is different from zero also if the representation conjugate to R_1 is obtained from the representation conjugate to R_2 by adding a box in the Young diagram.

Since $A = A_1 + A_2 = \alpha \log N + \beta$, this situation can occur in two ways:

(1) $A_2 = \beta_2$, $A_1 = \alpha \log N + \beta_1$: then $W \sim e^{-\frac{\beta_2}{2}}$,
(2) $A_1 = \alpha_1 \log N + \beta_1$, $A_2 = \alpha_2 \log N + \beta_2$ with $\alpha_1 > \alpha_2$: then $W \sim N^{-\frac{\alpha_2}{2}}$.

We proceed now to evaluate $W(A_1, A_2)$ in the phase before the cutoff, namely for $\alpha < 4$. The sum over representations in (51) can be replaced by the contribution of the dominant representation in the large N limit, calculated on the sphere in Section 3. The saddle point is unaffected by the presence of the extra term

$$\sum_i \prod_{j,j\neq i} \left(1 + \frac{1}{n_j - n_i}\right)\left(e^{\frac{A_1}{N} n_i} + e^{\frac{A_2}{N} n_i}\right)$$

which is subleading with respect to the action. After replacing in (51) the sum with the saddle point contribution, some simplifications occur and we get

$$W(A_1, A_2) = \frac{1}{N} \sum_i \prod_{j,j\neq i} \left(1 + \frac{1}{n_j - n_i}\right)\left(e^{\frac{A_1}{N} n_i} + e^{\frac{A_2}{N} n_i}\right). \tag{54}$$

The representation in (54) is of the type given in (15), and the sum over i describes all possible ways of adding a box to the Young diagram. However the replacement $n_i \to n_i + 1$ is impossible in the region $-l \leqslant i \leqslant l$ as the resulting sequence of integers would not be monotonic increasing. So the sum over i in (54) can be replaced by a sum over α with $1 \leqslant \alpha \leqslant l$. As a matter of fact we must consider only positive α's, as adding a box to the adjoint representation amounts to symmetrize with respect to A_1 and A_2, and it has been taken already into account. Hence in (54) we must replace the index i with α, n_i with $M - l + \alpha + r_\alpha$ while the index j goes from $-M$ to M, namely it goes over both the condensed and the non-condensed regions. With these substitutions the expression for the Wilson loop becomes:

$$\begin{aligned} W(A_1, A_2) = \sum_{\alpha=1}^{l} \frac{1}{\xi_\alpha}\left[1 - \frac{l}{N}\left(1 - \frac{\xi_\alpha}{l}\right)\right] \\ \times \exp\left(\sum_{\beta=1}^{l} \log\left(1 + \frac{1}{\xi_\alpha - \xi_\beta}\right) - \frac{A_2}{2}\left(1 - 2\frac{l - \xi_\alpha}{N}\right)\right) + \{A_2 \to A_1\}, \end{aligned} \tag{55}$$

where

$$\xi_\alpha = \alpha + r_\alpha. \tag{56}$$

It is convenient as usual to use in the large N limit the continuum variables $x = \frac{\alpha}{l}$, $\xi(x) = \frac{\xi_\alpha}{l}$ and the density function $\rho(\xi) = \frac{dx}{d\xi}$. The crucial part of the calculation is the evaluation of $\sum_{\beta=1}^{l} \log(1 + \frac{1}{\xi_\alpha - xi_\beta})$ which can be done following Ref. [29]. By expanding the logarithm one finds:

$$\sum_{\beta=1}^{l} \log\left(1 + \frac{1}{\xi_\alpha - \xi_\beta}\right) = \int d\eta\, \rho(\eta) \frac{1}{\xi - \eta} - \sum_{k=2} \frac{1}{k} \sum_\beta \frac{1}{(\beta - \alpha)^k} \rho(\xi)^k. \tag{57}$$

The first term at the r.h.s. comes from the $k = 1$ term of the log expansion and can be evaluated using Eq. (27), the other terms can be calculated as in [29] and give $\log \frac{\sin \pi\rho}{\pi\rho}$. By inserting these results into (55) and using the explicit form of the solution (30) one finally obtains (neglecting $O(\frac{l}{N})$ terms):

$$W(A_1, A_2) = 2\int d\xi \frac{\sin\pi\rho}{\pi}\left(e^{-\frac{A_2}{2}} + e^{-\frac{A_1}{2}}\right) = e^{-\frac{A_2}{2}} + e^{-\frac{A_1}{2}} \tag{58}$$

which is exactly the same result as in the frozen phase. The result is not trivial, but it was somehow to be expected. Both phases, before and after the transition, are strong coupling phases and the expectation value of the Wilson loop should be in both of them obtained, to the leading order, by filling the loop with elementary plaquettes in the fundamental representation. The effects of the transition are expected to appear only at the next-to-leading order ($\sim \frac{l}{N}$) which is sensitive to the $O(\frac{l}{N})$ degrees of freedom which are not frozen below the cutoff.

Acknowledgements

A.A. wants to thank the "Service de Physique Theorique (SPhT)" at Saclay for the kind hospitality of these last months. He also would like to thank I. Kostov and G. Vernizzi for many useful discussions during this period.

S.A. would like to thank the Niels Bohr Institute for warm hospitality during the early stage of this work.

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